Problem Set 11 – Relativity (GOI36A)

01/12/2020

1 The Reissner-Nordström Black Hole Solution

When we discuss "general relativity coupled to electromagnetism", what we mean concretely is that we want to consider the Einstein-Maxwell theory specified by the action

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) , \tag{1.1}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the field strength that corresponds to the photon field A_μ, and R is the usual Ricci scalar. As we discussed on a previous problem set, the equations of motion for this theory are

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) , \\ &\nabla_\mu F^{\mu\nu} = 0 . \end{aligned} \tag{1.2}$$

The first equation in (1.2) is the Einstein equation, where the stress-energy tensor corresponds to the photon being the matter field, while the second equation in (1.2) is the photon field equation of motion.

One solution to Einstein-Maxwell theory that we are keenly interested in is the electric Reissner-Nordström black hole, given by

$$\begin{aligned} ds^2 &= -\Delta(r) dt^2 + \Delta(r)^{-1} dr^2 + r^2 d\Omega^2 , \\ A_t &= \frac{Q}{\sqrt{4\pi} r} , \quad A_r = A_\theta = A_\phi = 0 , \end{aligned} \tag{1.3}$$

where we have defined the function $\Delta(r)$ by

$$\Delta(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} , \tag{1.4}$$

and we have set $c = 1$ for convenience. Note that this solution has all components of the field A_μ vanishing, except for the time component A_t. Another way to write this is through the use of form langauge, where we can specify the photon field as a one-form $A \equiv A_\mu dx^\mu$ such that

$$A = \frac{Q}{\sqrt{4\pi} r} dt . \tag{1.5}$$

These two different ways of specifying the photon field are equivalent, I just thought it might be helpful to see this form version explicitly.[1]

[1] You'll notice that our expression for the photon field has a slightly different normalization than you saw in class or in Carroll's book, due to the factor of $\sqrt{4\pi}$. This is due to our normalization conventions in the action being different. As long as we are careful about these conventions, though, the physics is all the same.

1.) Prove that the field strength tensor $F_{\mu\nu}$ has only two non-zero components:

$$F_{tr} = -F_{rt} = \frac{Q}{\sqrt{4\pi}r^2} \ , \tag{1.6}$$

while all other components of $F_{\mu\nu}$ vanish. Equivalently, we can write this in form language as

$$F = \frac{Q}{\sqrt{4\pi}r^2} dt \wedge dr \ , \tag{1.7}$$

where $F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ is the two-form version of the field strength tensor.

2.) Prove that this solution satisfies the equation of motion $\nabla_\mu F^{\mu\nu} = 0$ for the photon field A_μ. (Note: you will have to compute some Christoffel symbols for this problem, though certainly not the full set.)

3.) **Bonus:** pick a particular component of the Einstein equation, and prove that it vanishes. This of course is tedious, because you'll need to compute the Ricci scalar, which in turn requires knowing a lot of the components of the Riemann tensor. So, I'm leaving this optional, but at the very least make sure you understand the steps you would need to take in order to prove it.

4.) Write down explicitly what the Klein-Gordon equation $(\Box - m^2)\psi(x) = 0$ looks like on the Reissner-Nordström background, where $\Box \equiv \nabla_\mu \nabla^\mu$ and $\psi(x)$ is some scalar field. You may need to recall from a previous problem set that the Laplacian $\Box$ has an explicit formula of the form

$$\Box\psi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \psi \right) \ . \tag{1.8}$$

If this looks foreign to you, go back to problem set 8!

5.) Once you have written down the Klein-Gordon equation, prove that it is *separable*. That is, prove that you can make a careful separation of variables ansatz for ψ that results in a differential equation that only depends on the coordinate r and some constants.

In doing this, it may be helpful to recall that the spherical harmonics $Y_{\ell m}(\Omega)$ are eigenfunctions of the Laplacian on $\hat{L}^2$ on S^2, such that

$$\hat{L}^2 Y_{\ell m} = \ell(\ell+1) Y_{\ell m} \ , \quad \hat{L}^2 \equiv -\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta} \right) - \frac{1}{\sin^2\theta} \frac{d^2}{d\phi^2} \ . \tag{1.9}$$

2 One Fish, Two Fish, Redshift, Blueshift

In this problem, we will study how a photon propagating on a black hole background can undergo gravitational redshift and blueshift purely from gravitational effects alone.

6.) Consider two observers, Alice and Bob, who are at radial distances r_1 and r_2 (respectively) away from the center of a Reissner-Nordström black hole. Assume that both Alice and Bob are stationary and do not change their position, and that both are at the same fixed angular coordinate. Now, Alice uses a laser to shoot a photon to Bob. The photon initially has frequency f_1 when Alice first sends it out. Prove that when Bob receives the photon, its frequency f_2 as measured by Bob is now given by

$$f_2 = f_1 \frac{r_2}{r_1} \sqrt{\frac{r_1^2 - 2GMr_1 + GQ^2}{r_2^2 - 2GMr_2 + GQ^2}} \ . \tag{2.1}$$

7.) Suppose Alice is further away from the black hole than Bob (i.e. $r_1 > r_2$). Does the photon get redshifted (i.e. its frequency decreases) or blueshifted (i.e. its frequency increases) as it travels to Bob? What about if $r_2 > r_1$? Are your answers true for any values of Q and M, or do you get different cases for different regimes of the parameters? In particular, are there any subtleties in considering the Schwarzschild ($Q \to 0$) or extremal ($Q^2 \to GM^2$) limits of the metric?

8.) What if Bob is hanging out right at the black hole horizon? What happens to the frequency and energy of the photon as it gets to Bob? Should you be worried about your results?

3 The Penrose Diagram for the Reissner-Nordström Black Hole

Lastly, we'll spend some time in the problem session trying to make sense of the maximally-extended Penrose diagram for the Reissner-Nordström black hole, as seen for example on page 205 of https://arxiv.org/abs/gr-qc/9712019. You're more than welcome to take a look in advance and try to make sense of it on your own, but also feel free to wait for our group discussion in the problem session.